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# OPTIMAL STRENGTH DESIGN OF BEAM-COLUMNS

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**Abstract**—The paper considers the optimal design of elastic beam-columns which are acted upon simultaneously by an axial compressive and transverse forces. The design has to be such that it uses least amount of material to achieve prescribed strength under given loading conditions. Strength constraints are imposed to limit the intensity of maximum normal (flexural and axial) and shear stresses within the member. The optimal designs are also checked against serviceability and buckling requirements. Concepts from Differential Game Theory are employed for the solution of the optimization problem. Solutions are presented for beam-columns of different cross-sectional shapes and boundary conditions subjected to various combinations of axial and transverse forces.

## INTRODUCTION

Optimal design of structural elements that use the least amount of material to perform a given function has received considerable attention in the past two decades[1–10]. Several authors used classical variational techniques for solution of the optimization problems[4–10]. For instance, Karihaloo and Parbery[4–9] obtained optimal solutions for single purpose and multipurpose beam-columns for stiffness (serviceability) requirements. However, since from a practical viewpoint, the initial design is based on strength requirements limiting the maximum stresses to allowable values, it is important to obtain optimal strength solutions.

The present paper considers the optimal design of elastic beam-columns that use least amount of material to achieve prescribed strength under given loading conditions. Concepts from Differential Game Theory are employed for the solution of the problem. These concepts have been previously adopted to the optimization problems of statically determinate and indeterminate elastic members in flexure[1–3, 11]. The method is illustrated by considering a pin-ended beam-column under the simultaneous action of an axial compression and transverse distributed load. Strength constraints are specified to limit the intensity of maximum normal (flexural and axial) and shear stresses within the member. Optimal strength designs are checked against serviceability and buckling requirements. Solutions are also presented for beam-columns of different cross-sectional shapes and boundary conditions, subjected to various combinations of axial and transverse forces.

## 1. STATEMENT OF THE PROBLEM

Consider an elastic beam-column acted upon simultaneously by an axial compressive force $P^*$, and transverse forces $f^*(x^*)$ which cause bending moment $M_e^*(x^*)$ at a section $x^*$ of the member (Fig. 1). Taking the origin of co-ordinates at the left end of the member, the equilibrium equation can be written in the following dimensionless form:

$$\alpha^n y_{xx} + Py + M_e(x) = 0, \quad 0 \le x \le 1. \tag{1}$$

The geometric boundary conditions at the ends of the member must be specified for obtaining a solution of eqn (1). For instance, in case of a pinned beam-column these conditions are:

$$y(0) = 0, \quad y(1) = 0. \tag{2}$$

On the other hand, when mixed boundary conditions are specified, as in the case of a pinned-fixed beam-column considered later in the paper, eqn (1) is replaced by the more general equilibrium equation:

$$(\alpha^n y_{xx})_{xx} + P y_{xx} + f(x) = 0, \quad 0 \leq x \leq 1 \tag{3}$$

where $y = y^*/L$ is the transverse deflection and subscript $x$ denotes differentiation with respect to the dimensionless linear co-ordinate $x = x^*/L$, $L$ being the linear extent of the member, $P = P^*/EcL^{(2n-2)}$, $M_e(x) = M_e^*(x^*)/EcL^{(2n-1)}$ and $f(x) = f^*(x^*)/EcL^{(2n-3)}$, where $E$ is Young's modulus of the material and $c$ and $n$ are constants. It is assumed that the second moment of area $I(x^*)$ and the area of cross-section $A(x^*)$ are related through:

$$I = cA^n \tag{4}$$

where the constants $c$ and $n$ are determined by the cross-sectional shape. Thus $n = 1$ represents a sandwich cross-section or a rectangular cross-section of constant depth but varying width, $n = 2$—a geometrically similar cross-section (say, circular) and $n = 3$—a rectangular cross-section of constant width and varying depth. The non-dimensional cross-sectional area is defined by $\alpha(x) = A(x^*)/L^2$.

The strength (stress) constraints on the beam-column of a given cross-sectional form may be prescribed as

$$\Omega_i \equiv \sigma_i(x, s, f, y, M, Q, \alpha, \alpha_x) \leq 0 \quad (i = 1, 2) \tag{5}$$

where $x$ and $s$ are co-ordinates of the cross-sectional fibre at which the stress is being calculated, $\alpha(x)$ is the unknown cross-sectional area (the control) controlling the stress and $\alpha_x(x)$ its first derivative with respect to $x$; $M$ and $Q$ are the two main force resultants namely the bending moment and shear force which are related through

$$\frac{dM}{dx} = Q, \quad \frac{dQ}{dx} = -f(x), \tag{6}$$

where $f(x)$ is the external load at an arbitrary section $x$ along the member. The two stresses in (5) refer to normal and shear stress respectively. The normal stress has two components; one due to axial compression and the other due to flexure. Note that $M$ and $Q$ are dependent on $y$, which must be obtained in such a way that the equilibrium eqns (1) or (3) and the associated boundary conditions are satisfied.

The optimization problem consists in determining the control $\alpha(x)$ that meets the strength constraints (5) and minimizes the mass $W^*$ of the beam column

$$W^* = \gamma \int_0^L A(x^*)\, dx^* \tag{7}$$

where $\gamma$ is the mass density of the material. In dimensionless form the mass function is given by

$$W = \int_0^1 \alpha(x)\, dx \tag{8}$$

where $W = W^*/\gamma L^3$.

The optimization problem formulated above can be viewed as a differential game problem (game against Nature) whose solution can be sought by the minimax (or guaranteed) approach.

## 2. MINIMAX APPROACH

We indicate a method which allows us to reduce the solution of the game problem to the determination of the extrema of some variational problem. This method is based on the assumption that it is possible to obtain the dependence of elastic solutions on controls in explicit form. In other words, it is assumed that the force resultants (bending moment $M$ and shear force $Q$) can be found in a closed form $M = M(x, y, f, \alpha, \alpha_x)$ and $Q = Q(x, y, f, \alpha, \alpha_x)$.† Substituting the expressions for $M$ and $Q$ into the l.h.s. of the inequalities (5), we obtain

$$\Omega_i(x, s, f, y, \alpha, \alpha_x) \le 0, \quad (i = 1, 2) \tag{9}$$

where

$$\Omega_i(x, s, f, y, \alpha, \alpha_x) \equiv \sigma_i(x, s, f, M(x, y, f, \alpha, \alpha_x), Q(x, y, f, \alpha, \alpha_x), \alpha, \alpha_x).$$

The maxima of $\Omega_i(x, s, f, y, \alpha, \alpha_x)$ with respect to $s$ and $f$ are determined for fixed $x$ and $\alpha$. Assume that the maxima are attained for $s = s_i^*$ and $f = f_i^*$, i.e.

$$\Omega_i(x, s_i^*, f_i^*, y, \alpha, \alpha_x) \equiv \max_s \max_f \Omega_i(x, s, f, y, \alpha, \alpha_x) \tag{10}$$

and denote

$$\Omega_i^*(x, y, \alpha, \alpha_x) = \Omega_i(x, s_i^*(x, \alpha. \alpha_x), f_i^*(x, \alpha, \alpha_x), y, \alpha, \alpha_x). \tag{11}$$

Note that there is no need for maximization with respect to $f$ if the external loading is deterministic in nature. Making use of the notation (11), the strength constraints (9) can be written as

$$\Omega_i^*(x, y, \alpha, \alpha_x) \le 0. \tag{12}$$

When the deflection $y$ is obtained as a function of $x$ from eqns (1) or (3) and the associated boundary conditions, the strength constraints may be formally re-written as

$$\Omega_i^*(x, \alpha, \alpha_x) \le 0. \tag{13}$$

Thus, the initial problem (1)–(8) reduces to a variational problem of minimizing with respect to $\alpha$ the integral (8) under the differential constraints (13) and any other conditions imposed on the function $\alpha(x)$.

## 3. ILLUSTRATIVE EXAMPLE

Consider the beam-column shown in Fig. 1 which is subjected to a general loading system in $xy$-plane. Let the locations of the concentrated forces $W_i^*$ be designated by $l_i$ and that of moments $M_i^*$ by $c_i$ both measured from the left end of the member. For clarity of presentation, the asterisks are omitted in the sequel.

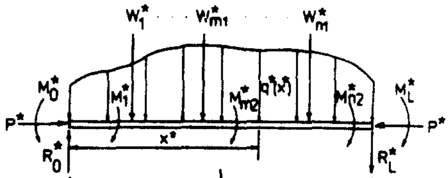


Fig. 1. Beam-column with general loading.

†In determinate beam-column structures, $M$ and $Q$, like the desired control $\alpha$, are expressible as function of $x$, $f$ and $y$ alone. However, in indeterminate beam-column structures the bending moment and shear force depend not only on $x$, $f$ and $y$ but also on the control $\alpha$ through the compatibility (deformation) requirement. In general, the expressions for $M$ and $Q$ involve as many unknowns as the degree of redundancy of the structure. The unknowns may be support reactions and/or the locations of points of inflexion; additional conditions for determining these unknowns, together with $\alpha$, follow from compatibility requirements.

Let the number of concentrated forces and moments acting between $x = 0$ and the section $x$ be $m_1$ and $m_2$ respectively. Then the expressions for shear force $Q$ and bending moment $M$ at section $x$ are

$$Q = R_0 - \sum_{i=1}^{m_1} W_i - \int_0^x f(\eta)\,\mathrm{d}\eta + P\frac{\mathrm{d}y(x)}{\mathrm{d}x} \tag{14}$$

$$M = R_0 x - M_0 - \int_0^x f(\eta)(x-\eta)\,\mathrm{d}\eta - \sum_{i=1}^{m_1} W_i(x - l_i) + \sum_{i=1}^{m_2} M_i + Py(x), \tag{15}$$

where $f(x)$ is the intensity of any lateral distributed loading and $R_0$ the terminal shear force at $x = 0$. Note that both $M$ and $Q$ are functions of the design $\alpha(x)$ through the unknown $y(x)$. For a given set of loads $f(x)$, $W_i$, $M_i$ and $P$, the deflection of the member can be found by Galerkin's method[12].

The intensity of the normal stress (due to bending moment and axial force) $\sigma_x$ and the transverse (shear) stress $\tau_{xy}$ at any cross-section $x$ of the beam-column must satisfy the following strength constraints:

$$\begin{aligned} \sigma_1 &\equiv |\sigma_x| - \sigma_0 \leq 0, \\ \sigma_2 &\equiv |\tau_{xy}| - \tau_0 \leq 0, \end{aligned} \tag{16}$$

where $\sigma_0$ and $\tau_0$ are prescribed positive constants. Sometimes it is convenient to specify the following strength criterion

$$(\sigma_x^2 + \alpha\tau_{xy}^2)^{1/2} - k \leq 0. \tag{17}$$

If $\alpha = 4$, it restricts the maximum shear stress and for $\alpha = 3$ the value of the strain energy. Note that if $\alpha = 4$ and $\sigma_0 = 2\tau_0 = k$, (17) is identical to (16).

From elastic beam-column theory, $\sigma_x$ and $\tau_{xy}$ are given by

$$\sigma_x = \frac{Ms(x)}{I(x)} + \frac{P}{A(x)}; \quad \tau_{xy} = K\frac{\partial}{\partial x}\left(\frac{Ma(x,s)}{I(x)}\right) \tag{18}$$

where $s(x)$ defines the position of a cross-sectional fibre, and $K$ and $a(x, s)$ are determined by the cross-sectional shape. For the mass-stiffness relationship (4) assumed in the present paper, the following expressions result:

For $n = 1$, $I(x) = b(x)h^3/12$ ($h$ = constant depth, $b(x)$ = variable width), $K = 1/b(x)$, $a(x, s) = b(x)((h^2/4) - s^2)/2$ and $-h/2 \leq s \leq h/2$; for $n = 2$, $I(x) = \pi r^4(x)/4$ (where $r(x)$ is variable linear dimension - radius of a circular section), $K = 1/[3(r^2(x) - s^2)^{1/2}]$, $a(x, s) = (r^2(x) - s^2)^{3/2}$ and $-r(x) \leq s \leq r(x)$; and for $n = 3$, $I(x) = bh^3(x)/12$, $K = 1/b$, $a(x, s) = (b/2)((h^2(x)/4) - s^2)$ and $-h(x)/2 \leq s \leq h(x)/2$.

We now derive optimal solutions for various values of $n$ appearing in the mass-stiffness relationship (4).

### 3.1 n = 1 (*Rectangular cross-section of constant depth,* h, *and variable width* b(x))

Strength constraints (16)', together with (18), take the following form

$$\begin{aligned} \sigma_1 &\equiv \frac{1}{b(x)h}\left|\frac{12Ms}{h^2} + P\right| - \sigma_0 \leq 0, \\ \sigma_2 &\equiv \frac{6}{b(x)h^3}\left(\frac{h^2}{4} - s^2\right)|Q| - \tau_0 \leq 0. \end{aligned} \tag{19}$$

Since $\sigma_1$ and $\sigma_2$ attain their maximum values at $s = h/2$ and $s = 0$, respectively, the inequalities

(19) can be written as

$$\frac{1}{b(x)h}\left|\frac{6M}{h}+P\right|-\sigma_0\le 0,$$
$$\frac{3|Q|}{2b(x)h}-\tau_0\le 0. \tag{20}$$

The function $b(x)$ is to be determined in such a way that the inequalities (20) are satisfied and the total mass $W^* = \gamma h\int_0^L b(x)\,dx$ is minimized. The solution follows directly from (20)

$$b(x)\ge \max\left\{\frac{|6M+Ph|}{\sigma_0 h^2},\frac{3|Q|}{2h\tau_0}\right\}. \tag{21}$$

For example, consider a pin-ended beam-column subject to a uniformly distributed lateral load of intensity $w$ and an axial compression $P$. The optimal solution for one half of the member is given in the following dimensionless form

$$\alpha(x)=\begin{cases} B(1-2x)+Cy_x; & 0\le x\le x_1\\ Dx(1-x)+Fy+G; & x_1\le x\le 1/2\end{cases} \tag{22}$$

where $\alpha(x)=b(x)h/L^2$, $B=3w/4\tau_0L$, $C=3P/2\tau_0L^2$, $D=3w/h\sigma_0$, $F=6P/hL\sigma_0$, $G=P/\sigma_0L^2$. Note that $y(x)$, and hence $\alpha(x)$ is obtained by an iterative procedure from (1) and (22) using Galerkin's method.

3.2 n = 2 (*Geometrically similar sections say, circular*)

Although the solution procedure is the same for beam-columns of any geometrically similar cross-sectional shape, the method is illustrated with respect to a beam-column of circular cross-section. With the aid of (18) the inequalities (16) become

$$\sigma_1\equiv\frac{1}{\pi r^2(x)}\left|\frac{4Ms}{r^2(x)}+P\right|-\sigma_0\le 0, \tag{23}$$

$$\sigma_2\equiv\frac{4}{3\pi r^4(x)}\left|Q(r^2(x)-s^2)+M\frac{dr}{dx}\left(\frac{4s^2}{r(x)}-r(x)\right)\right|-\tau_0\le 0. \tag{24}$$

In order to evaluate the maxima of $\sigma_1$ and $\sigma_2$ we note that $\sigma_1$ attains its maximum at $s=r(x)$ and $\sigma_2$ either at $s^2=0$ or at $s^2=r^2(x)$. Hence it follows that

$$\frac{1}{\pi r^2(x)}\left|\frac{4M}{r(x)}+P\right|-\sigma_0\le 0, \tag{25}$$

$$\max(\psi_1,\psi_2)-\tau_0\le 0, \tag{26}$$

where

$$\psi_1=\frac{4}{3\pi r^3(x)}\left|Qr(x)-M\frac{dr}{dx}\right|, \tag{27}$$

$$\psi_2=\frac{4}{\pi r^3(x)}\left|M\frac{dr}{dx}\right|. \tag{28}$$

From (26)–(28) the following differential inequalities result

$$\frac{dr}{dx}\le\frac{1}{M}\min\{Qr(x)+3\beta r^3(x),\beta r^3(x)\}, \tag{29}$$

$$\frac{dr}{dx} \geq \frac{1}{M} \max\{Qr(x) - 3\beta r^3(x), -\beta r^3(x)\}, \tag{30}$$

where $\beta = \pi\tau_0/4$.

When $Q$ is positive, the solution domain can be divided into two sub-domains $s_1(r(x) \leq \sqrt{Q/2\beta})$ and $s_2(r(x) \geq \sqrt{Q/2\beta})$. In these sub-domains the differential inequalities (29), (30) can be written as

$$\frac{1}{M}[Qr(x) - 3\beta r^3(x)] \leq \frac{dr}{dx} \leq \frac{\beta r^3(x)}{M} \quad (r(x) \epsilon s_1) \tag{31}$$

$$-\frac{\beta r^3(x)}{M} \leq \frac{dr}{dx} \leq \frac{\beta r^3(x)}{M} \quad (r(x) \epsilon s_2). \tag{32}$$

It should be mentioned that the inequalities (32) are consistent for any value of $r(x) \epsilon s_2$, but that the inequalities (31) can be solved if

$$r(x) \geq \sqrt{\frac{Q}{4\beta}}. \tag{33}$$

From (31) and (32) it follows that the admissible functions $r(x)$ must satisfy the differential inequality

$$\frac{dr}{dx} \geq \frac{1}{M}[Qr(x) - 3\beta r^3(x)], \tag{34}$$

subject to the solvability condition (33).

Proceeding along similar lines for negative $Q$, we find the admissible functions $r(x)$ must satisfy the differential inequality

$$\frac{dr}{dx} \leq \frac{1}{M}[Qr(x) + 3\beta r^3(x)] \tag{35}$$

subject again to the solvability condition (33). The actual optimal design $r(x)$ is chosen by studying the behaviour of admissible curves $r(x)$ obtained as a result of solving the differential inequalities (34) and (35).

The optimization problem therefore reduces to determining the $r(x)$ that satisfies (25), (33) and (34) or (35) and minimizes the total mass (8) of the beam-column.

Again, consider the example of a pin-ended beam-column acted upon by a uniformly distributed load of intensity $w$ and axial compression $P$. The optimal solution in dimensionless form is given by

$$\alpha(x) = \begin{cases} F_2; & 0 \leq x \leq x_1 \\ F_1; & x_1 \leq x \leq 1/2 \end{cases} \tag{36}$$

where

$$F_1 = [\{B + \sqrt{B^2 - C}\}^{1/3} + \{B - \sqrt{B^2 - C}\}^{1/3}]^2 \pi^{1/3}, (B^2 - C > 0)$$

$$F_2 = \frac{4[Dx(1-x) + Gy(x)]^2}{Dx^2(3 - 2x) + 6G\int_0^x y(\eta)\,d\eta},$$

$B = Hx(1-x) + Sy(x)$, $C = P^3/27\pi\sigma_0^3L^6$, $D = w/2\tau_0L$, $G = P/\tau_0L^2$, $H = w/\sigma_0L$, $S = 2P/\sigma_0L^2$, and $F_2(0) = \frac{4}{3}[D + Gy_x(0)]$.

The unknown deflection $y(x)$ and hence $\alpha(x)$ are obtained by an iterative procedure from (1) and (36) involving Galerkin's technique.

3.3 n = 3 (*Rectangular cross-section with constant width* b *and variable depth* h(x))

Following along lines similar to those for $n = 2$, it is found that the optimal depth function $h(x)$ must satisfy the following algebraic and differential inequalities

$$h(x) \geq [P + (P^2 + 24\, bM\sigma_0)^{1/2}]/2b\sigma_0, \tag{37}$$

$$\frac{dh(x)}{dx} \geq h(x)[Q - 2\lambda h(x)]/M, \tag{38}$$

$$h \leq Q/3\lambda, \tag{39}$$

where $\lambda = b\tau_0/3$.

In the example of the pin-ended beam-column acted upon by a uniformly distributed load $w$ and an axial compressive load $P$, the non-dimensional optimal control takes the following form:

$$\alpha(x) = \begin{cases} B(1-x) + Cy(x)/x; & 0 \leq x \leq x_1 \\ D + [D^2 + Fx(1-x) + Gy(x)]^{1/2}; & x_1 \leq x \leq 1/2 \end{cases} \tag{40}$$

where $B = 3w/4\tau_0 L$, $C = 3P/2\tau_0 L^2$, $D = P/2\sigma_0 L^2$, $F = 3bw/4\sigma_0 L^2$, $G = 3bP/2\sigma_0 L^3$. Note that $\alpha(0) = B + Cy_x(0)$.

The unknowns $y(x)$ and $\alpha(x)$ are obtained from (1) and (40) by an iterative procedure using Galerkin's method.

In writing the optimal solution (22), (36), (40) it was assumed that shear stresses govern the design near the supports and normal stresses near the midspan. This will occur if $G < B + Cy_x(0)$ when $n = 1$, $B < 2F_2(0)$ when $n = 2$ and $D < 0.5\ [B + Cy_x(0)]$ when $n = 3$. The transition point $x_1$ is evaluated by equating the two expressions for $\alpha(x)$ in eqns (22), (36) and (40) and then solving for $x_1$. On the other hand, when the axial load is large, it is likely that the design of the whole span is governed by the normal stresses alone. It must be mentioned that because of symmetry, it is sufficient to consider only one half of the beam-column; the boundary conditions (2) being replaced by

$$y(0) = y_x(1/2) = 0. \tag{41}$$

At this stage, it is worth noting that the optimal designs (22), (36) and (40) are characterized by several common features. For instance, in all the designs the maximum shear stress $\tau_m$ achieves the value $\tau_0$ in the region $0 \leq x \leq x_1$, whereas the maximum normal stress $\sigma_m$ in any cross-section does not reach the respective allowable value $\sigma_0$. In the region $x_1 \leq x \leq 1/2$, $\sigma_m = \sigma_0$ but $\tau_m < \tau_0$. However, at the transition point $x_1$ the stresses achieve the maximum permissible values, i.e. $\sigma_m = \sigma_0$ and $\tau_m = \tau_0$. Because of symmetry, the characteristics are the same for the other half of the beam-column. Thus, the optimal beam-column is, in general, divided into several regions in which shear and normal stresses alternately govern the design.

## 4. ADDITIONAL EXAMPLES

Optimal solutions for a pin-ended beam-column subjected to other transverse loading conditions are quoted below in the following dimensionless form:

$$\alpha(x) = \max\{F_1(x), F_2(x)\}. \tag{42}$$

The effects of axial and transverse forces are clearly distinguished in the way the expressions for $F_1(x)$ and $F_2(x)$ are written below.

4.1 *Beam-column subjected to a concentrated load* Q *at* x = d n = 1

$$F_1(x) = \begin{cases} (1-d)B + Cy_x; & 0 \leq x \leq d^- \\ Bd - Cy_x & ; \ d^+ \leq x \leq 1 \end{cases} \tag{43}$$

$$F_2(x) = \begin{cases} D(1-d)x + Gy + H; & 0 \le x \le d \\ Dd(1-x) + Gy + H; & d \le x \le 1, \end{cases} \tag{44}$$

where $B = 3Q/2\tau_0 L^2$, $C = 3P/2\tau_0 L^2$, $D = 6Q/\sigma_0 hL$, $G = 6P/\sigma_0 hL$, $H = P/\sigma_0 L^2$.

<u>$n = 2$</u>

$$F_1(x) = \begin{cases} \dfrac{2[B(1-d)x + Cy(x)]^2}{3\left[\dfrac{B}{2}(1-d)x^2 + C\displaystyle\int_0^x y(\eta)\,d\eta\right]}; & 0 \le x \le d^- \\[2ex] \dfrac{2[Bd(1-x) + Cy(x)]^2}{3\left[\dfrac{B}{2}d(1-x)^2 + C\displaystyle\int_x^1 y(\eta)\,d\eta\right]}; & d^+ \le x \le 1 \end{cases} \tag{45}$$

$$F_2(x) = [(D + \sqrt{D^2 - G})^{1/3} + (D - \sqrt{D^2 - G})^{1/3}]^2 \pi^{1/3}; \quad 0 \le x \le 1 \tag{46}$$

$$F_1(0) = \frac{4}{3}[B(1-d) + Cy_x(0)] \tag{47}$$

$$F_1(1) = \frac{4}{3}[Bd - Cy_x(1)], \tag{48}$$

where

$$D = \begin{cases} H(1-d)x + Sy(x); & 0 \le x \le d, \\ Hd(1-x) + Sy(x); & d \le x \le 1, \end{cases}$$

$B = Q/\tau_0 L^2$, $C = P/\tau_0 L^2$, $G = P^3/27\pi\sigma_0^3 L^6$, $H = 2Q/\sigma_0 L^2$, $S = 2P/\sigma_0 L^2$, and $(D^2 - G) > 0$.

<u>$n = 3$</u>

$$F_1(x) = \begin{cases} B(1-d) + Cy(x)/x; & 0 \le x \le d^- \\ Bd + Cy(x)/(1-x); & d^+ \le x \le 1 \end{cases} \tag{49}$$

$$F_2(x) = \begin{matrix} D + [D^2 + G(1-d)x + Hy(x)]^{1/2}; & 0 \le x \le d \\ D + [D^2 + Gd(1-x) + Hy(x)]^{1/2}; & d \le x \le 1 \end{matrix} \tag{50}$$

$$F_1(0) = B(1-d) + Cy_x(0), \tag{51}$$

$$F_1(1) = Bd - Cy_x(1), \tag{52}$$

where $B = 3Q/2\tau_0 L^2$, $C = 3P/2\tau_0 L^2$, $D = P/2\sigma_0 L^2$, $G = 3bQ/2\sigma_0 L^3$, $H = 3bP/2\sigma_0 L^3$.

### 4.2 *Beam-column subjected to an end moment* $M_0$ *at* $x = 0$

<u>$n = 1$</u>

$$F_1(x) = B - Cy_x, \tag{53}$$

$$F_2(x) = D(1-x) + Gy(x) + H, \tag{54}$$

where $B = 3M_0/2\tau_0 L^3$, $C = 3P/2\tau_0 L^2$, $D = 6M_0/\sigma_0 hL^2$, $G = 6P/\sigma_0 hL$, $H = P/\sigma_0 L^2$.

n = 2

$$F_1(x) = \frac{4[B(1-x)+Cy(x)]^2}{3\left[B(1-x)^2+2C\int_x^1 y(\eta)\,d\eta\right]}; \quad 0 \le x \le 1, \tag{55}$$

$$F_2(x) = [(D+\sqrt{D^2-G})^{1/3}+(D-\sqrt{D^2-G})^{1/3}]^2\pi^{1/3}; \quad 0 \le x \le 1 \tag{56}$$

$$F_1(1) = \frac{4}{3}[B - Cy_x(1)] \tag{57}$$

where $D = H(1-x)+Sy(x)$, $B = M_0/\tau_0L^3$, $C = P/\tau_0L^2$, $G = P^3/27\pi\sigma_0^3L^6$, $H = 2M_0/\sigma_0L^3$, $S = 2P/\sigma_0L^2$, and $(D^2-G)>0$.

n = 3

$$F_1(x) = B + Cy(x)/(1-x); \quad 0 \le x \le 1 \tag{58}$$

$$F_2(x) = D+\sqrt{D^2+G(1-x)+Hy(x)}; \quad 0 \le x \le 1 \tag{59}$$

$$F_1(1) = B - Cy_x(1), \tag{60}$$

where $B = 3M_0/2\tau_0L^3$, $C = 3P/2\tau_0L^2$, $D = P/2\sigma_0L^2$, $G = 3M_0b/2\sigma_0L^4$, $H = 3Pb/2\sigma_0L^3$.

### 4.3 *A pinned-fixed beam-column subjected to a distributed load* w *(Fig. 2)*

n = 1

$$F_1(x) = |-B + C(1-2x)+Dy_x|, \tag{61}$$

$$F_2(x) = |-Gx + Hx(1-x)+Ry(x)| + S, \tag{62}$$

where $B = 3M_L/2\tau_0L^3$, $C = 3w/4\tau_0L$, $D = 3P/2\tau_0L^2$, $G = 6M_L/\sigma_0hL^2$, $H = 3w/\sigma_0h$, $R = 6P/\sigma_0hL$, $S = P/\sigma_0L^2$.

n = 2

$$F_1(x) = \begin{cases} \dfrac{2}{3}\dfrac{G(x)^2}{\int_0^x G(\eta)\,d\eta}; & 0 \le x \le x_0 \\ \dfrac{2}{3}\dfrac{G(x)^2}{\int_x^1 G(\eta)\,d\eta}; & x_0 \le x \le 1, \end{cases} \tag{63}$$

$$F_2(x) = \begin{cases} [(G_1+\sqrt{G_1^2-H})^{1/3}+(G_1-\sqrt{G_1^2-H})^{1/3}]^2\pi^{1/3}; & (G_1^2-H)>0 \\ U\cos^2\left[\frac{1}{3}(\cos^{-1} q/p^{3/2})\right]; & (G_1^2-H)<0, \end{cases} \tag{64}$$

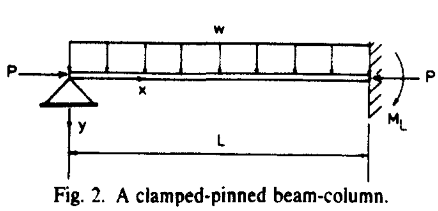

Fig. 2. A clamped-pinned beam-column.

$$F_1(0) = \frac{4}{3}[B - C + Dy_x(0)] \tag{65}$$

where $G(x) = Bx(1-x) - Cx + Dy(x)$, $G_1(x) = Sx(1-x) - Rx + Ty(x)$, $B = w/2\tau_0 L$, $C = M_L/\tau_0 L^3$, $D = P/\tau_0 L^2$, $H = P^3/27\pi\sigma_0^3 L^6$, $R = 2M_L/\sigma_0 L^3$, $S = w/\sigma_0 L$, $T = 2P/\sigma_0 L^2$, $U = 4\pi p/L^2$, $p = P/3\pi\sigma_0$, $q = L^3 G_1(x)/\pi$, and $\bar{x}$ is the point of inflexion.

$\underline{n = 3}$

$$F_1(x) = \begin{cases} -B + C(1-x) + Dy(x)/x; & 0 < x \le x_0 \\ |-Bx + Cx(1-x) + Dy(x)|/|x - \bar{x}|; & x_0 \le x \le 1 \end{cases} \tag{66}$$

$$F_2(x) = G + [|-Rx + Sx(1-x) + Ty(x)| + G^2]^{1/2}, \quad 0 \le x \le 1 \tag{67}$$

$$F_1(0) = |-B + C + Dy_x(0)|, \tag{68}$$

where $B = 3M_L/2\tau_0 L^3$, $C = 3w/4L\tau_0$, $D = 3P/2L^2\tau_0$, $G = P/2\sigma_0 L^2$, $R = 3bM_L/2\sigma_0 L^4$, $S = 3bw/4\sigma_0 L^2$, $T = 3bP/2\sigma_0 L^3$, and $\bar{x}$ is the point of inflexion.

Computational aspects of the solutions are briefly discussed in the next section.

## 5. COMPUTATIONAL SCHEME

In the Galerkin's method the deflection function is assumed and the error in the Galerkin's integrals[12] is minimized. It should be noted that not only $y$ but also its higher derivatives appearing within the Galerkin's integrals are to be adequately represented by the trial function. Having chosen a suitable trial function for deflection, the residues in the Galerkin's integrals are evaluated by using appropriate expressions for $\alpha(x)$. The procedure is repeated with a different deflection profile until all the residues are smaller than a prescribed quantity. Newton–Raphson technique was used for rapid convergence of the solution.

With the exception of the pinned-fixed beam-column (Fig. 2), equilibrium eqn (1) was used for solution by the above procedure. In the case of a pinned-fixed beam-column the fourth order eqn (3) had to be used because of the mixed boundary conditions,

$$y(0) = 0,\ y(1) = 0,\ y_x(1) = 0,\ \alpha^n y_{xx}|_{x=1} = M_1, \tag{69}$$

where $M_1 = M_L/EcL^{(2n-1)}$. In this context it should be pointed out that solutions by Galerkin's method using the fourth order equation may not converge to the true solution. This situation arose when the above examples were solved using the fourth order eqn (3) and boundary conditions and the solutions compared with the results obtained by using second order equation in Galerkin's technique and by direct numerical integration of the differential eqn (3). This is probably because of the higher derivatives involved in using eqn (3) which affects the convergence of Galerkin's method[12]. Therefore the following iterative procedure was adopted to solve the pinned-fixed beam-column, shown in Fig. 2.

For a given problem, the dimensionless fixed end moment $M_1$ is assumed in the first iteration ($i = 1$). Also a regular function $y_{xx} = -1 (0 \le x \le 1)$ is assumed in the first iteration ($j = 1$) within the inner loop (steps iii–vi).

(i) Assume the redundant $(M_1)_i$

(ii) $(y_{xx})_i = -.1.0$

(iii) $(y_x)_j = \int_1^x y_{\eta\eta}\, d\eta$

(iv) $y_j = \int_0^x y_\eta\, d\eta$

(v) evaluate $\alpha_j(x)$

(vi) $(y_{xx})_j = -[Cy(x) + Bx(1-x) - (M_1)_i x]/\alpha_j^n$ where $C = P/EcL^{(2n-2)}$, $B = w/EcL^{(2n-3)}$, and $M_1 = M_L/EcL^{(2n-1)}$.

(vii) Repeat steps (iii) to (vi) if $|(y_{xx})_{i+1} - (y_{xx})_i| > 10^{-6}$

(viii) Repeat steps (ii)–(vii) if $y_i(1) > 10^{-5}$, with $(M_1)_{i+1} = (M_1)_i + \Delta$, where $\Delta$ is obtained by Newton–Raphson technique.

Numerical examples are considered in the next section.

## 6. NUMERICAL EXAMPLES AND DISCUSSION

The optimal designs for various values of $n$ are shown in Figs. 3–11 for pin-ended beam-columns and in Figs. 12–14 for pinned-fixed beam-columns. The loading, geometrical and material properties used in these examples are indicated on the figures. The regions of the optimal design governed by the limiting transverse and normal stresses are clearly distinguished. Thus the profile for normal stress constraint alone is shown by dotted lines, the profile for transverse stress constraint alone by broken lines and finally, the optimal profile for both the constraints by solid lines.

From a practical viewpoint, besides the strength requirements, the structural members are required to meet serviceability requirements. In general, the maximum deflection of a structural member must be less than the allowable value stipulated in the Codes of Practice and be safe against buckling. All the numerical examples considered above were checked against these requirements.

In order to judge the material saving made possible by optimization, the volume of the optimal designs was compared with that of prismatic beam-columns which satisfied the same strength requirements as the optimal beam-columns. The volume of such beam-columns, together with the percentage savings made possible by optimization, is shown in Table 1. It is clear that optimization leads to significantly lighter designs.

The paper has only briefly explored the applications of concepts from Differential Game Theory to strength optimization of beam-columns. The method can also be applied to beam-columns with other boundary conditions where they form a part of multiple span frames. These results will be reported in future communications.

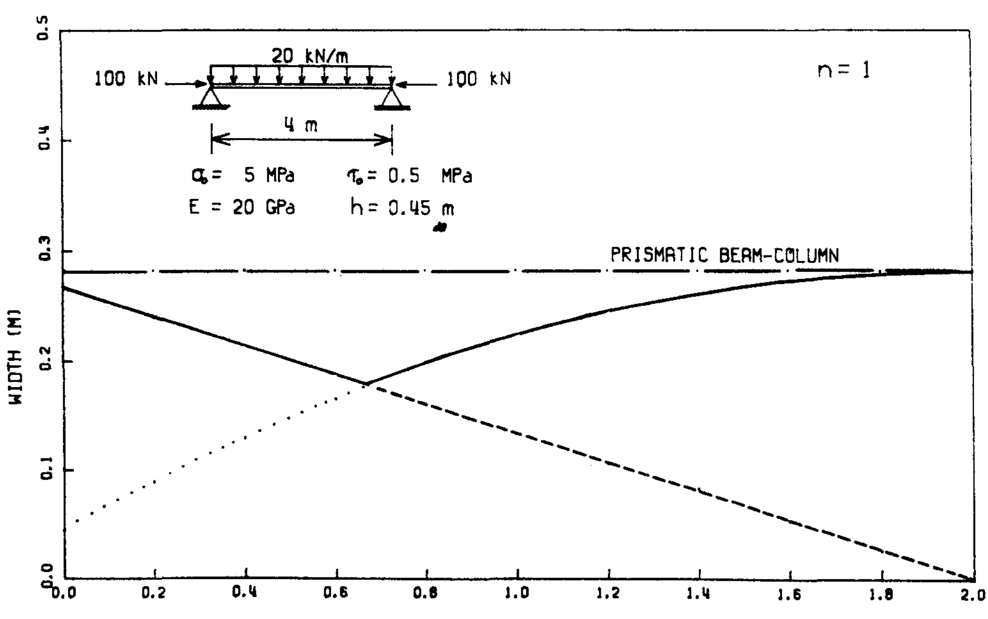


Fig. 3. Optimal design of a simply supported beam-column subject to a uniformly distributed lateral load and an axial compression for a solid rectangular cross-section of constant depth and variable width, $n = 1$.

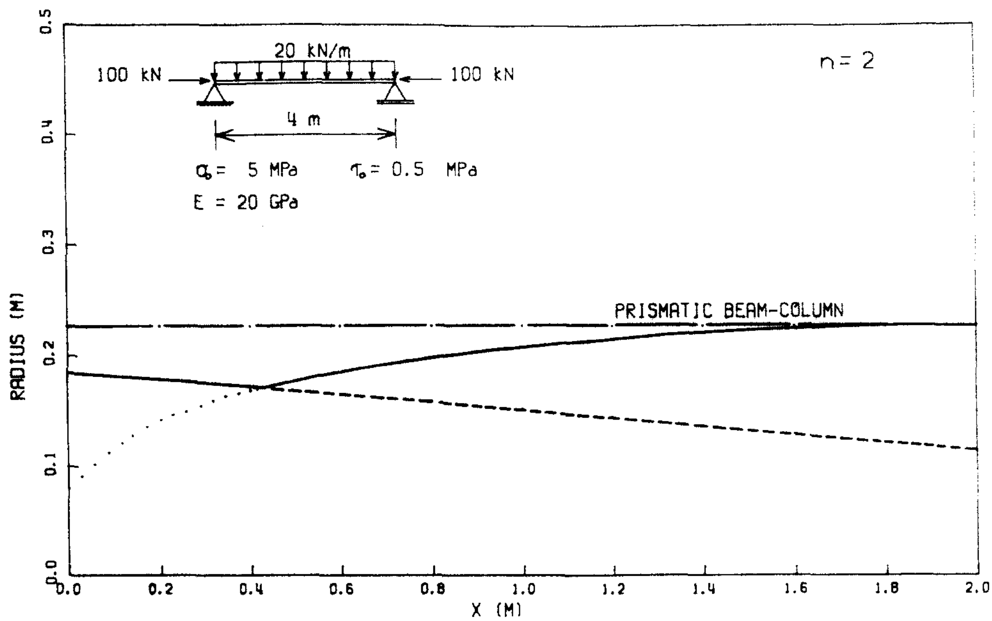


Fig. 4. Optimal design of the structure shown in Fig. 3 for a solid circular cross-section, $n = 2$.

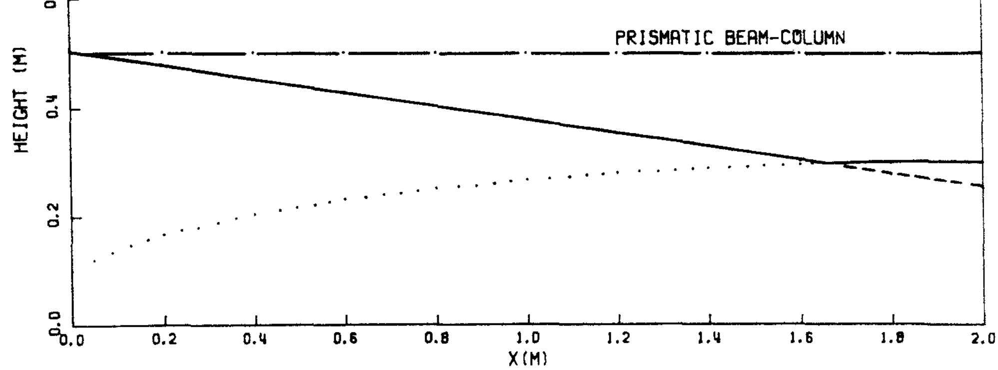


Fig. 5. Optimal design of the structure shown in Fig. 3 for a solid rectangular cross-section of constant width and variable depth, $n = 3$.

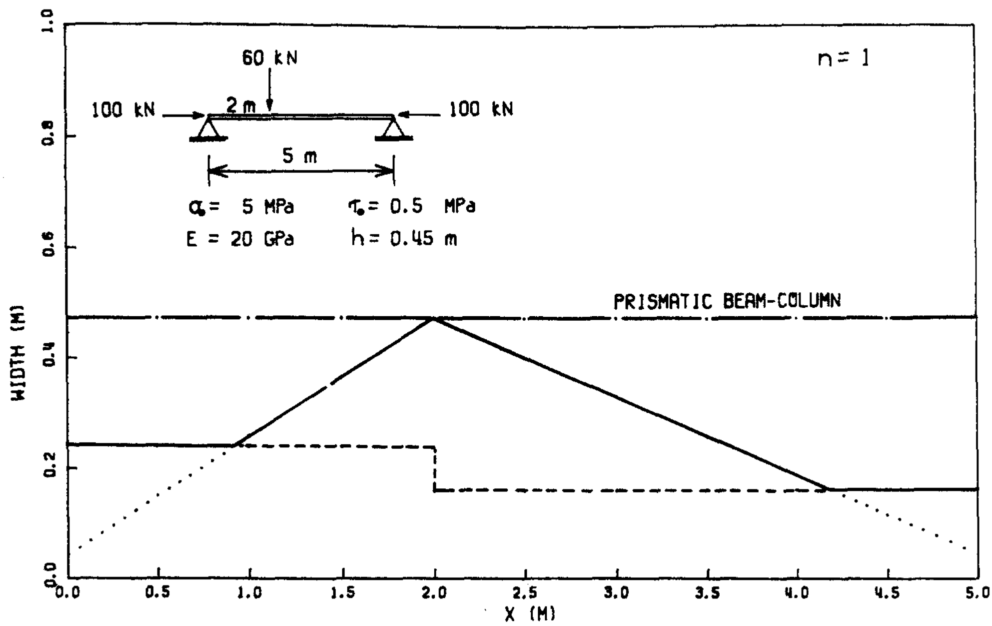


Fig. 6. Optimal design of a simply supported beam-column subject to a concentrated lateral load and an axial compression for a solid rectangular cross-section of constant depth and variable width, $n = 1$.

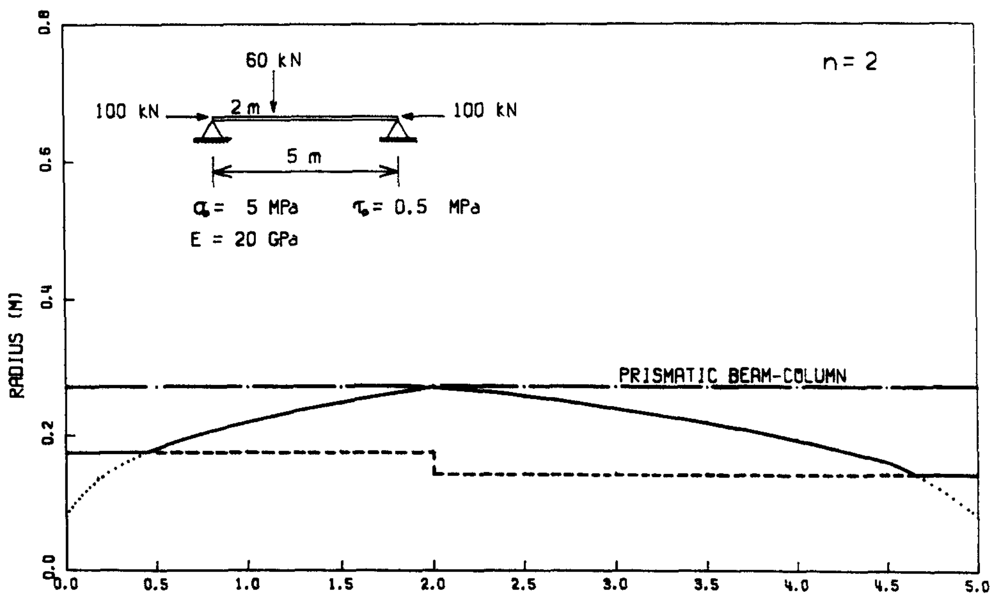


Fig. 7. Optimal design of the structure shown in Fig. 6 for a solid circular cross-section, $n = 2$.

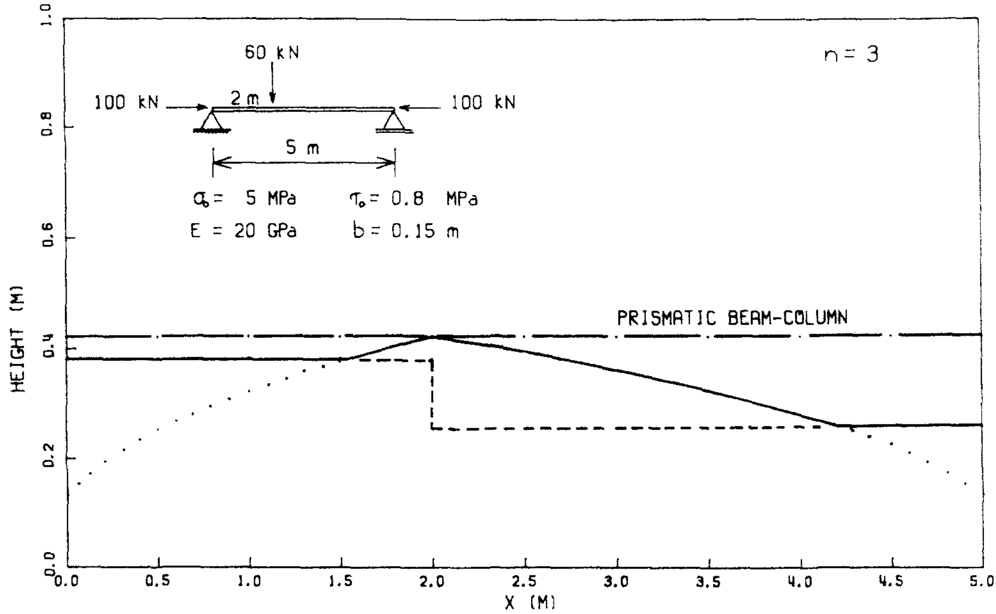


Fig. 8. Optimal design of the structure shown in Fig. 6 for a rectangular cross-section of constant width and variable depth, $n = 3$.

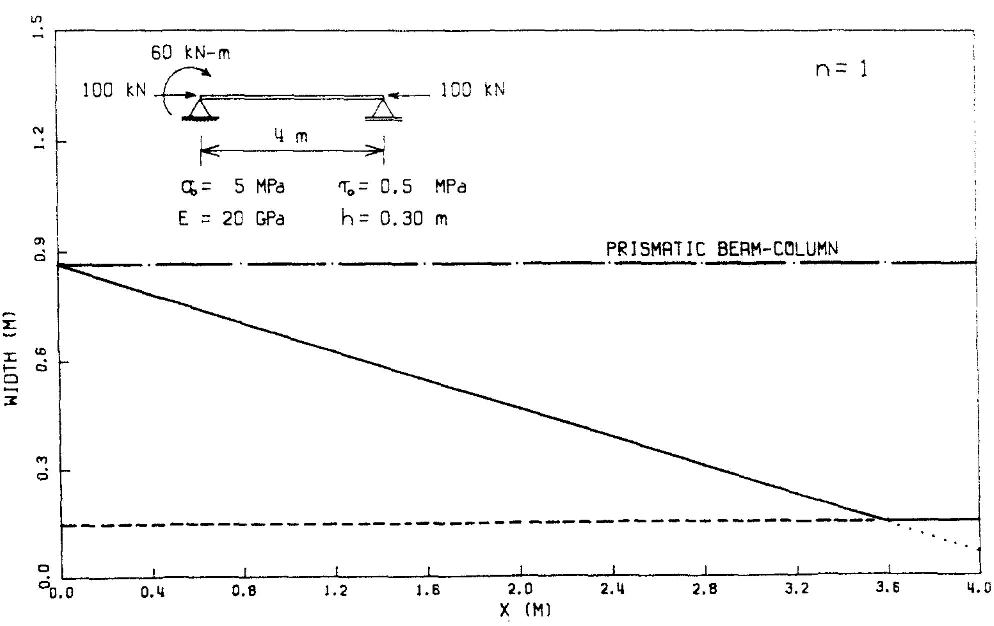


Fig. 9. Optimal design of a simply supported beam-column subject to an end moment and an axial compression for a solid rectangular cross-section of constant depth and variable width, $n = 1$.

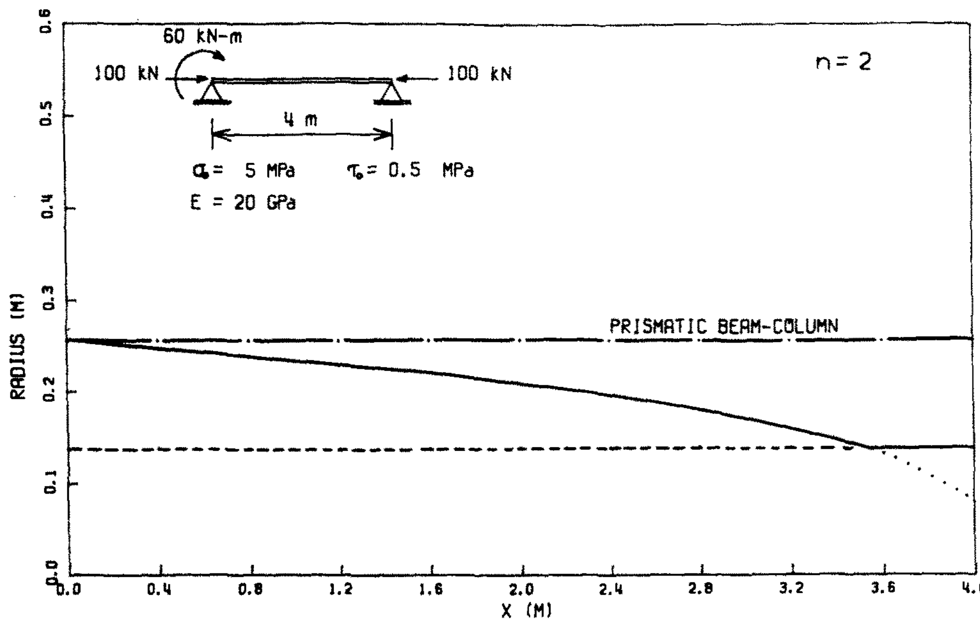


Fig. 10. Optimal design of the structure shown in Fig. 9 for a solid circular cross-section, $n = 2$.

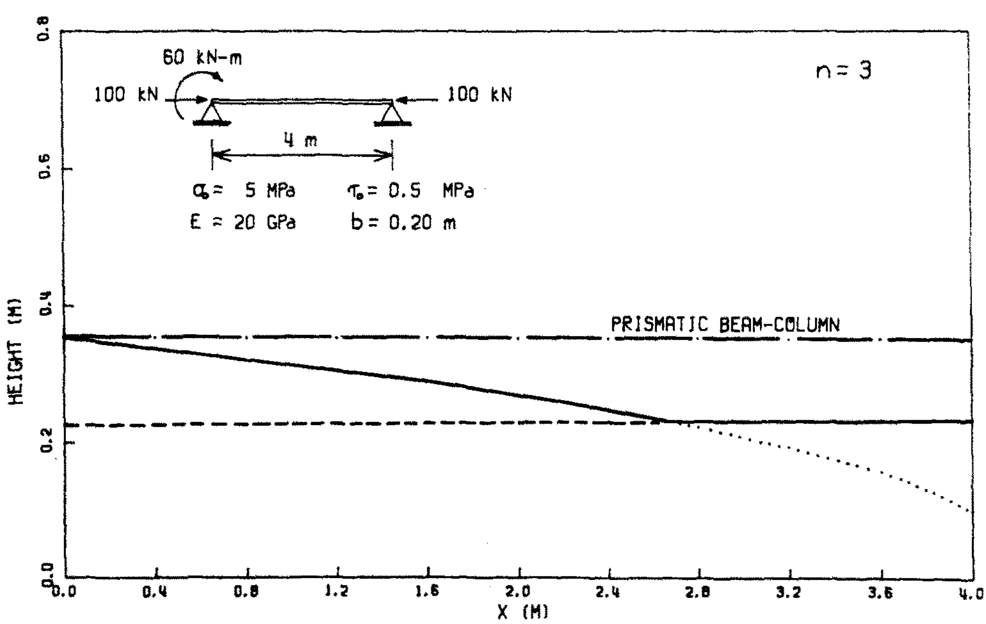


Fig. 11. Optimal design of the structure shown in Fig. 9 for a solid rectangular cross-section of constant width and variable depth, $n = 3$.

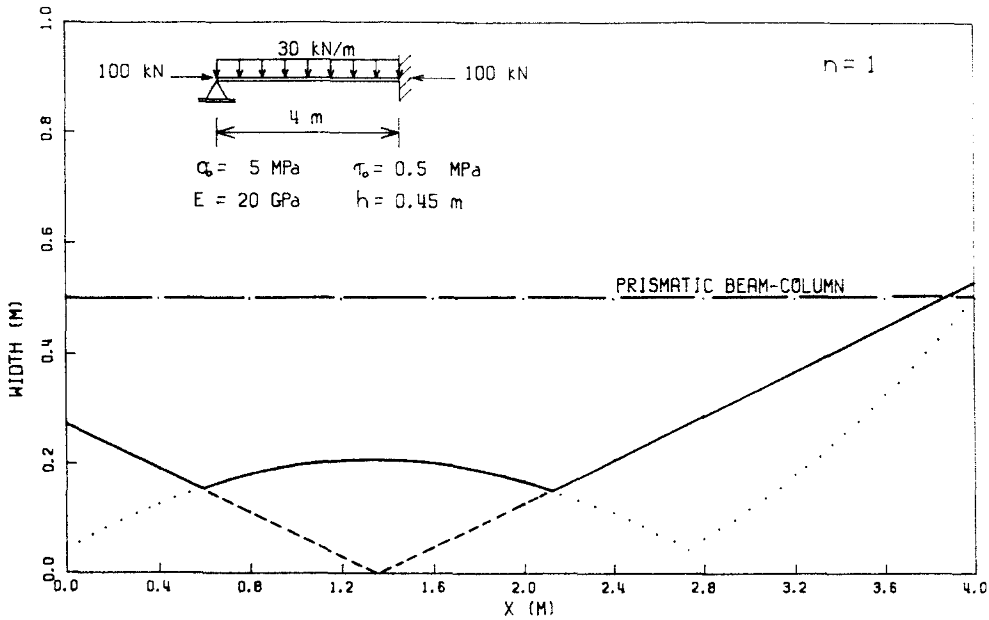


Fig. 12. Optimal design of a clamped-pinned beam-column subject to a uniformly distributed lateral load and an axial compression for a solid rectangular cross-section of constant depth and variable width, $n = 1$.

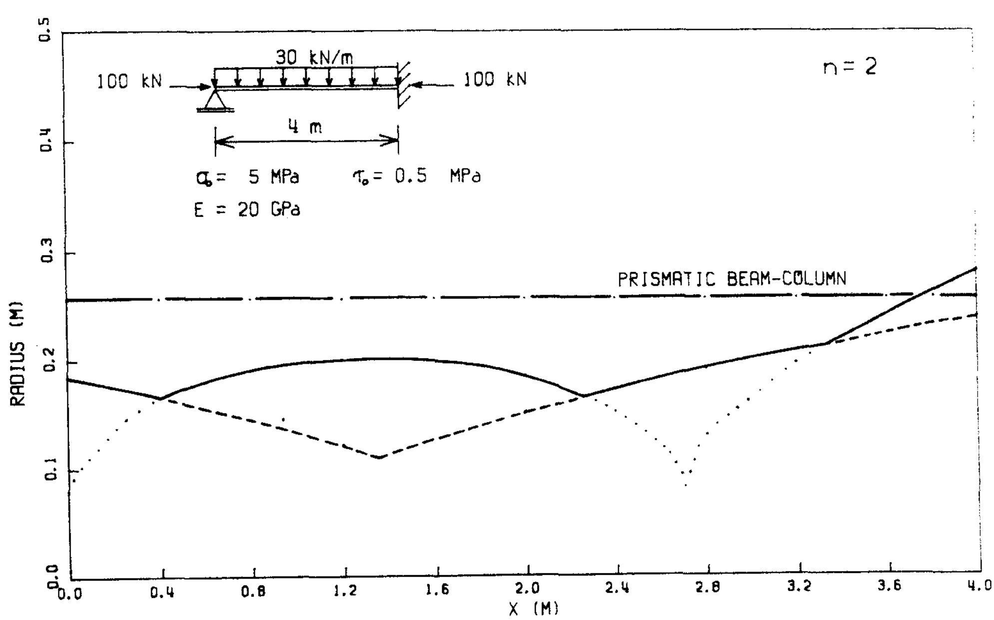


Fig. 13. Optimal design of the structure shown in Fig. 12 for a solid circular cross-section, $n = 2$.

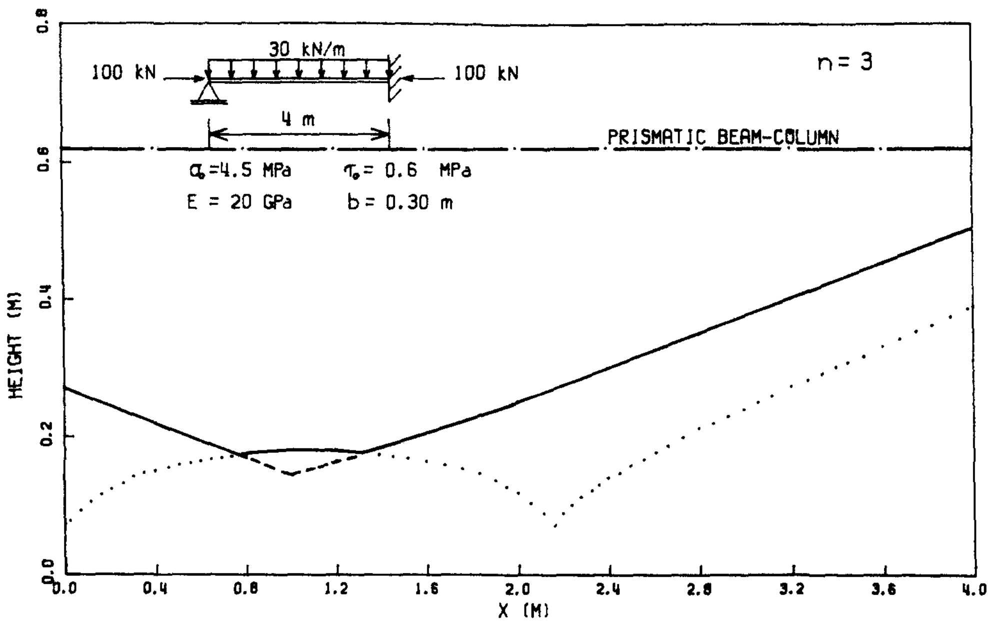


Fig. 14. Optimal design of the structure shown in Fig. 12 for a solid rectangular cross-section of constant width and variable depth, $n = 3$.

Table 1. Comparison of volume ($m^3$) of the optimal designs shown in Figs. 3 and 14 with corresponding prismatic beam-columns

| Beam Shape | Fig. 3 | Fig. 4 | Fig. 5 | Fig. 6 | Fig. 7 | Fig. 8 | Fig. 9 | Fig. 10 | Fig. 11 | Fig. 12 | Fig. 13 | Fig. 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Optimal | 0.43 | 0.52 | 0.31 | 0.65 | 0.75 | 0.26 | 0.57 | 0.53 | 0.22 | 0.48 | 0.50 | 0.35 |
| Prismatic | 0.51 | 0.65 | 0.40 | 1.06 | 1.16 | 0.32 | 1.04 | 0.83 | 0.28 | 0.90 | 0.83 | 0.74 |
| % Saving | 16 | 20 | 22 | 39 | 35 | 19 | 45 | 36 | 21 | 47 | 40 | 53 |

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